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A PROBLEM OF ERDŐS ON ABELIAN GROUPS

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The following theorem is proved. If $a_1, a_2, \ldots, a_n$ are nonzero elements in $\mathbf{Z}_n$, and are not all equal, then $\varepsilon_1 a_1+\varepsilon_2 a_2+\ldots+\varepsilon_n a_n=0$ has at least n solutions with $\varepsilon_i=0$ or 1.

1.

P. Erdős has asked (personal letter from R. L. Graham): If $a_1, a_2, \ldots, a_p$ are nonzero elements of $\mathbf{Z}_p$ (p prime), not all equal, is it true that

$$\varepsilon_1 a_1+\varepsilon_2 a_2+\ldots+\varepsilon_p a_p = 0$$

has at least p solutions $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_p)$ of the form $\varepsilon_i=0$ or 1?

We shall prove that the answer is yes. Further, p need not be prime; the result carries over just as well with an arbitrary finite abelian group in place of $\mathbf{Z}_p$.

Theorem 1. *If $a_1, a_2, \ldots, a_n$ is a sequence of n nonzero elements, not all equal, in an abelian group of order n, then*

$$\varepsilon_1 a_1+\varepsilon_2 a_2+\ldots+\varepsilon_n a_n = 0 \tag{1}$$

has at least n solutions $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ of the form $\varepsilon_i=0$ or 1.

2.

Our proof will be based on the following combinatorial lemma.

Lemma. *Let A be a matrix with entries 0 or 1 of size k by n satisfying:*

(i) $1\leq k\leq n-2$.

(ii) *Every row contains at least one* 0.

(iii) *Every row contains at least two* 1*'s.*

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Then one can permute the columns of A so that, in the resulting matrix, every row takes the form

$$\ldots 1 \ldots 0 \ldots 1 \ldots;$$

in other words, in every row, the 1's do not form a consecutive block.

Proof. We proceed by induction on k. The lemma is clearly true for $k=1$. Taking $k \geqq 2$, we assume the lemma true for matrices with $k-1$ rows.

We call an entry of the matrix *scarce* if either it is the 0 in a row having only one 0, or it is a 1 in a row having exactly two 1's. Since $n \geqq 4$, every row contains at most two scarce entries. Hence the matrix itself contains at most $2k$ scarce entries. Since $n > k$, it follows that some column contains at most one scarce entry. By permuting rows and columns, we may arrange the matrix so that its first column contains no scarce entry except possibly its entry in the first row. Thus we may assume the matrix has the form

$$A = \left[\begin{array}{c|c} a & \cdots \\ \hline \vdots & B \end{array}\right]$$

where the $k-1$ by $n-1$ submatrix B has at least one 0 and at least two 1's in each of its rows. By induction, we may permute columns 2 through n of A so that every row of B has the form $\ldots 1 \ldots 0 \ldots 1 \ldots$.

If $a=1$, we are done unless the first row of A has the form $11 \ldots 1\ 0 \ldots 0$ (a block of two or more 1's followed by all 0's). But then, we need only move the first column of A over to the right of the nth column.

If $a=0$, we may break up any block of 1's in the first row by moving the first column to the right to a position between some two other columns. ∎

Proof of Theorem 1. Throughout, $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ will denote a vector with components $\varepsilon_i = 0$ or 1. We show first that there is a nontrivial solution to (1)—a result which Erdős and Graham [2, p. 95] attribute to prehistoric man. If the n partial sums $a_1, a_1+a_2, \ldots, a_1+a_2+\ldots+a_n$ are distinct, then 0 must be among them. If some two are the same: $a_1+\ldots+a_i = a_1+\ldots+a_j$ $(i<j)$, then $a_{i+1}+a_{i+2}+\ldots$ $\ldots+a_j=0$. Thus not only is there a nontrivial solution to (1), but *there is a nontrivial solution in which the 1's among $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ form a consecutive block.* We consider two cases.

Case 1. $a_1+a_2+\ldots+a_n \neq 0$. Having found one nontrivial solution to (1), we assume that we have found $k \geqq 1$ nontrivial solutions. Let A be the k by n matrix having these solutions as its rows. Every row of A has at least one 0 since $a_1+a_2+\ldots$ $\ldots+a_n \neq 0$. Every row has at least two 1's since, by hypothesis, the a_i are nonzero. Assuming that $k \leqq n-2$, we may apply the lemma to the matrix A. Thus, in effect, we may rearrange the sequence $a_1, a_2, \ldots, a_n$ so that, in each of our k nontrivial solutions to (1), the 1's do not form a consecutive block. But, as we have seen, there is, for any ordering, a solution to (1) in which the 1's do form a consecutive block. Therefore there is an additional nontrivial solution to (1). This proves that there are at least $n-1$ nontrivial solutions to (1); hence there are at least n solutions in all.

Case 2. $a_1+a_2+\ldots+a_n=0$. In this case, for each solution $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ to (1), its "complement" $1-\varepsilon_1, 1-\varepsilon_2, \ldots, 1-\varepsilon_n$ is also a solution. We shall assume that $n \geqq 5$, for Theorem 1 is easy to verify (in the case under consideration) for the group of order 3 and the two groups of order 4.

Suppose the terms a_i take only the values x and $-x$. Thus $a_i = x$ for $u \geqq 1$ indices i and $a_j = -x$ for $v \geqq 1$ indices j, where $u+v=n$. Hence Hence $a_i + a_j = 0$ for $uv \geqq n-1$ pairs $i \neq j$. This accounts for $n-1$ nontrivial solutions to (1), and we are done.

Thus we may assume that $a_i \neq \pm a_j$ for some pair i, j. We may therefore rearrange the sequence so that $a_{n-1} \neq \pm a_n$. Further, *if it is possible* to do so, we choose the last two terms so that the equation

$$\varepsilon_1 a_1 + \ldots + \varepsilon_{n-2} a_{n-2} + a_{n-1} + a_n = 0 \tag{2}$$

has a solution with $\varepsilon_1, \ldots, \varepsilon_{n-2}$ not all 1.

We show next that *there is a solution to*

$$\varepsilon_1 a_1 + \ldots + \varepsilon_{n-2} a_{n-2} \in \{0, -a_{n-1}, -a_n\} \tag{3}$$

with $\varepsilon_i = 0$ *or* 1, *in which the* 1's *among* $\varepsilon_1, \ldots, \varepsilon_{n-2}$ *form a consecutive block of length* s, $2 \leqq s \leqq n-3$. To see this, we consider the $n-2$ partial sums

$$a_1, a_1 + a_2, \ldots, a_1 + a_2 + \ldots + a_{n-2}. \tag{4}$$

If 0 is among these, or if two are the same, then

$$a_{j+1} + a_{j+2} + \ldots + a_{j+s} = 0,$$

for some $1 \leqq j+1 \leqq j+s \leqq n-2$. Now $s \geqq 2$ since the a_i are nonzero. Also $a_1 + \ldots + a_{n-2} \neq 0$, because $a_1 + \ldots + a_n = 0$ and $a_{n-1} + a_n \neq 0$; hence $s \leqq n-3$. If the $n-2$ sums in (4) are distinct and nonzero, then either $-a_{n-1}$ or $-a_n$ must occur among them. The sum $a_1 + \ldots + a_{n-2}$ is different from both $-a_{n-1}$ and $-a_n$ since $a_1 + \ldots + a_n = 0$. Also $a_1 = -a_{n-1}$ implies $a_2 + \ldots + a_{n-2} = -a_n$, and $a_1 = -a_n$ implies $a_2 + \ldots + a_{n-2} = -a_{n-1}$. Since $n \geqq 5$, we have, in any case, a solution to (3) with the 1's forming a consecutive block of length s, and $2 \leqq s \leqq n-3$.

Now suppose we have found $k \geqq 1$ solutions $(\varepsilon_1, \ldots, \varepsilon_{n-2})$ to (3), each containing at least one 0 and at least two 1's. We may apply the lemma to the k by $n-2$ matrix whose row vectors are these k solutions. Assuming that $k \leqq n-4$, we may, by the lemma, rearrange the subscripts on $a_1, \ldots, a_{n-2}$ so that, in each of our k solutions, the 1's do not form a consecutive block. Hence there is an additional solution to (3) having at least one 0 and at least two 1's. This proves that there are at least $n-3$ such solutions to (3).

In summary, we have found $n-3$ solutions to (1) satisfying

$$2 \leqq \varepsilon_1 + \ldots + \varepsilon_{n-2} \leqq n-3, \quad 0 \leqq \varepsilon_{n-1} + \varepsilon_n \leqq 1. \tag{5}$$

There are also the two extreme solutions $\varepsilon_1 = \ldots = \varepsilon_n = 0$ and $\varepsilon_1 = \ldots = \varepsilon_n = 1$. We need to find one more solution to (1).

If there is a nontrivial solution to (1) having $\varepsilon_{n-1} = \varepsilon_n = 0$, then the complement $1-\varepsilon_1, \ldots, 1-\varepsilon_n$ is a solution not among those we have accounted for, and we are done. Therefore, we may assume that all solutions $\varepsilon_1, \ldots, \varepsilon_n$ to (1) that satisfy (5) have $\varepsilon_{n-1} + \varepsilon_n = 1$. It follows that $\varepsilon_1 + \ldots + \varepsilon_n \geqq 3$, for all solutions to (1) that satisfy (5). Hence we may assume that $a_i \neq -a_j$ for $i \neq j$.

If there is a solution to (2) other than $\varepsilon_1=\ldots=\varepsilon_{n-2}=1$, we are done. Therefore, going back to the choice of a_{n-1} and a_n, we may assume that it was impossible to arrange the terms so that $a_{n-1}\neq\pm a_n$ and still have a solution to (2) with $\varepsilon_1, \ldots, \varepsilon_{n-2}$ not all 1.

It follows that if $\varepsilon_1, \ldots, \varepsilon_n$ is any solution to (1) satisfying $1\leqq\varepsilon_1+\ldots+\varepsilon_n\leqq n-1$, then the terms a_i for which $\varepsilon_i=1$ must all have the same value, say x. Also, by considering the complement $1-\varepsilon_1, \ldots, 1-\varepsilon_n$, we conclude that all terms a_i for which $\varepsilon_i=0$ have the same value y. Since $x\neq y$ (by hypothesis), it follows that there are exactly two solutions to (1) satisfying $1\leqq\varepsilon_1+\ldots+\varepsilon_n\leqq n-1$. But there are $n-3$ solutions to (1) satisfying (5), hence $n=5$. But $sx=(n-s)y=0$, $s=\varepsilon_1+\ldots+\varepsilon_n$, cannot hold in the group of order $n=5$. With this contradiction the proof is complete. ∎

As a consequence of Theorem 1, we have the following result.

Corollary 1.1. *Suppose $a_1, a_2, \ldots, a_n$ is a sequence of n nonzero elements, not all equal to $\pm a_1$, in an abelian group G of order n, and $g\in G$. If the equation*

$$\varepsilon_1 a_1+\varepsilon_2 a_2+\ldots+\varepsilon_n a_n = g \tag{6}$$

has one solution $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ of the form $\varepsilon_i=0$ or 1, then it has at least n such solutions.

Proof. Assuming that $g\neq 0$ and that there is a solution to (6), we may arrange the notation so that $a_1+a_2+\ldots+a_t=g$. Thus (6) is equivalent to

$$(\varepsilon_1-1)a_1+\ldots+(\varepsilon_t-1)a_t+\varepsilon_{t+1}a_{t+1}+\ldots+\varepsilon_n a_n = 0. \tag{7}$$

But (7) has at least n solutions with $\varepsilon_i=0$ or 1 since the sequence $-a_1, \ldots, -a_t, a_{t+1}, \ldots, a_n$ satisfies the hypothesis of Theorem 1. ∎

3.

Finally, we should point out that Theorem 1 is of interest only for the cyclic group, where it is sharp. For noncyclic abelian groups we may obtain stronger results by a quite different argument as follows.

Let G be a finite abelian group of order n. There is a smallest number s_0 such that, for *every* sequence $g_1, g_2, \ldots, g_s$ in G with $s\geqq s_0$, some subsequence has sum 0. The number s_0 depends on the structure of G. Now assume that $a_1, a_2, \ldots, a_s$ is a sequence in G and $s\geqq s_0$. By a result of Olson [3, Theorem 2], the equation

$$\varepsilon_1 a_1+\varepsilon_2 a_2+\ldots+\varepsilon_s a_s = 0 \tag{8}$$

has at least 2^{1+s-s_0} solutions (with $\varepsilon_i=0$ or 1). By a result of Eggleton and Erdős [1, Theorem 3], if G is a noncyclic abelian group of order n, then $s_0\leqq 1+n/2$. Hence if we take $s=n$, as in Theorem 1, equation (8) has at least $2^{n/2}$ solutions. This proves

Theorem 2. *If $a_1, a_2, \ldots, a_n$ is a sequence of length n in a noncyclic abelian group of order n, then*

$$\varepsilon_1 a_1+\varepsilon_2 a_2+\ldots+\varepsilon_n a_n = 0$$

has at least $2^{n/2}$ solutions $(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ of the form $\varepsilon_i=0$ or 1. ∎

Surprisingly, Theorem 2 and, more significantly, the two quoted results used in its proof, carry over to non-abelian groups. The proofs require considerable modification of the abelian versions. These results will be presented in a future paper on the $s_0=s(G)$ problem in non-abelian groups.

References

[1] R. B. EGGLETON and P. ERDŐS, Two combinatorial problems in group theory, *Acta Arith.* **21** (1972), 111—116.

[2] P. ERDŐS and R. L. GRAHAM, *Old and New Results in Combinatorial Number Theory*, Monographie No. 28 de L'Enseignement Mathématique, Genève, 1980.

[3] J. E. OLSON, A combinatorial problem on finite Abelian groups, II, *J. Number Theory* **1** (1969), 195—199.

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